Stochastic modeling of time series radar interferometry

Ramon F. Hanssen
Delft Institute of Earth Observation and Space Systems
Delft University of Technology
Delft, The Netherlands
Email: hanssen@geo.tudelft.nl

***Abstract*— Quality description and evaluation of InSAR results is hampered by the fact that the model to derive parameters from the observations is usually underdetermined. Only using strong, often rather qualitative, assumptions it is possible to reach unique solutions. One of the most prominent assumptions is that phase ambiguity resolution can be treated as a deterministic problem. In this study, a model formulation is presented that captures the majority of the assumptions in a mathematical sense, allowing for adjustment, testing procedures and formal error propagation. The influence of stochastic ambiguity resolution to the probability distribution of the estimated parameters is shown.**

I. Introduction

InSAR models to estimate topography or deformation from either one SAR pair or from time series usually rely on a number of rather harsh assumptions. Assumptions that are often encountered, and often not explicitly stated, are that (i) an interferogram only contains topography, or only deformation, (ii) that coherence estimators to describe the quality of a single resolution cell using a spatial or temporal estimation window are reliable and robust, (iii) that ambiguity resolution (phase unwrapping) can be treated as a deterministic image or data processing problem, using heuristic assumptions on maximum allowable phase gradients, and (iv) that error sources such as those caused by the atmosphere can be treated in a descriptive, qualitative way. Even though these assumptions are narrowed down in the most recent InSAR methods, they are often still the basis for applied studies.

As a result, quality estimates of the first information-bearing derived observables are often inadequate. To illustrate this point, consider the situation of one repeat-pass interferogram, spanning time interval Δt and perpendicular baseline $B_{\perp}$. Given two randomly selected resolution cells, there are two relevant parameters; (i) a topographic height difference between the points and (ii) a differential deformation. Now it should be possible to express the accuracy and reliability of these estimated parameters. Although there have been attempts to derive these quantities, it is in principle impossible to do so, since the problem is underdetermined, with two principal parameters and only one (derived) observation. This problem is often circumvented by assumptions, e.g. that there is no deformation, or that topography is known. However, there are a number of other problems involved as well. One of these problems is that it is often assumed that the phases are unwrapped, that is, the ambiguities per resolution cell have been resolved. Until now, this has been treated as a deterministic image processing problem, a trick that can be performed in a preprocessing stage. It is clear to realize that this approach may work, in that it produces intuitively appealing results, but the simple fact that it is an estimation problem and that errors in the ambiguity resolution can be made demonstrates that this stochasticity should be included in the parameter estimation problem.

In this paper we present a formulation of an alternative problem formulation in an attempt to formalize potential assumptions in a stochastic sense. This way, also the a priori information that may be needed to derive estimates is subject to hypothesis testing, making precision and reliability claims more robust.

II. Primary and derived variates

InSAR measurements are a showcase example of derived variates or observables. Here we distinguish the primary or intermediate variates and the 'information-bearing' variates.

The primary variates are the raw reflected radar pulses, involving in-phase and quadrature phase observations as a function of time. After A/D conversion and telemetry, a (phase-preserving) focusing algorithm is used to create a single-look complex (SLC) image. Thus, the next intermediate variates are the I/Q values per resolution cell. Although the amplitude of these complex values contains information, the geometric information in the phase is completely masked by the scattering component of the phase. Creating an interferogram from two SLC images produces complex (interferometric) values at each resolution cell. However, even these variates are still intermediate since they do not contain relevant geometric information. For example, total atmospheric delay adds up to several meters, orbit accuracies are in the (sub)-decimeter level, and the total number of integer phase cycles is not known, making (sub)-centimeter precision claims impossible. The first information-bearing variate is therefore the phase difference between two interferometric resolution cells. For repeat-pass interferometry we define the double-difference observation as the phase difference between two

0-7803-8743-0/04/$20.00 (C) 2004 IEEE

positions, differenced in time:

$$\begin{aligned}\varphi_{12}^{12} &= W\{\Big(W\{(\phi_1^1 - \phi_1^2)\} - W\{(\phi_2^1 - \phi_2^2)\}\Big)\} \\ &= (\phi_1^1 - \phi_1^2) - (\phi_2^1 - \phi_2^2) - a\end{aligned} \tag{1}$$

where the subscripts indicate the pixel positions 1 and 2 and the superscripts the time of acquisition, $t_1 = 1$, $t_2 = 2$. Expressing phase observations in cycles, the wrapping operator $W\{\zeta\} = \mod(\zeta + \frac{1}{2}) - \frac{1}{2} = \zeta - a$, with $a \in$ and $W\{\zeta\} \in [-\frac{1}{2}, \frac{1}{2}) \subset$. The functional relation between this variate and the unknown parameters is

$$E\{\varphi_{12}^{12}\} = k_1\, D + k_2\, H - a \tag{2}$$

where

- φ_{12}^{12} is the double-difference observable, expressed in cycles,
- k_1 $= 2\lambda^{-1}$,
- k_2 $= k_1\, B_\perp\, \cos^{-1}\theta\, R^{-1}$,
- D [m] is the unknown double-difference deformation vector component in line-of-sight,
- H [m] is the unknown topographic height difference between points 1 and 2, and
- a is the unknown double-difference integer cycle ambiguity [$a \in$].

and $B_\perp$, θ, R, D, H, and a are short hand notations for ${B_\perp}_{12}^{12}$, θ_{12}^{12}, R_{12}^{12}, D_{12}^{12}, H_{12}, and a_{12}^{12}, respectively. The expectation operator $E\{\cdot\}$ implies that the double-difference error sources, such as decorrelation and atmospheric signal or residual orbital signal, have zero-expectation. These are further modeled in the stochastic model, see below.

It is evident that it is mathematically impossible to solve for the three unknown parameters only from one double-difference observation. However, this is exactly what is happening in many of the studies using some form of InSAR (conventional 2-pass, 3-pass, 2-pass+DEM, Persistent scatterer interferometry (PSI), interferogram stacks). In most of these cases, information used for this interpretation and for making quality statements stems either from the qualitative (visual) interpretation of the interferogram or data (*'the fringes look smooth'* or *'the time series fit to the model very well'*), or from a-priori knowledge on the roughness of the topography, the type of deformation (abruptly or gradual), and the assumption that the ambiguity resolution is error-free.

The main proposition in this study is that if it is possible to make a 'human' visual interpretation of an interferogram or a time series it should be possible to capture this in terms of the mathematical model. In other words, if two pixels would be randomly selected from an interferogram, it should be possible to produce best estimates of the parameters of interest and give an optimal quality description of these parameters, preferably in terms of its multivariate probability density function (pdf).

III. Gauss Markov model

Here we chose to use the generic Gauss-Markov model as our starting point. Using eq. (2) we design the functional and stochastic model in matrix form as

$$E\{\begin{bmatrix}\varphi_{12}^{12} \\ d \\ h\end{bmatrix}\} = \begin{bmatrix}k_1 & k_2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}D \\ H \\ a\end{bmatrix}, \quad \begin{matrix}(D, H \in \quad) \\ (a \in \quad)\end{matrix} \tag{3}$$

and

$$D\{\begin{bmatrix}\varphi_{12}^{12} \\ d \\ h\end{bmatrix}\} = \begin{bmatrix}\sigma_\varphi^2 & & \\ & \sigma_d^2 & \\ & & \sigma_h^2\end{bmatrix}, \tag{4}$$

respectively. In this formulation we added pseudo-observations for double-difference deformation d and height difference h, short for d_{12}^{12} and h_{12}, respectively, and their respective variances. Note that this model is easy to expand for including more points and more interferograms, hereby allowing for all forms of interferometry. The main contribution of this formulation is that it allows for estimating the best linear unbiased estimator of the parameter vector, its full variance-covariance matrix, and even its multi-variate pdf. The fact that the integer ambiguities are included in the parameter vector makes the stochastic nature of its estimators explicit.

To be suitable for estimation, one of the main challenges is now to include all human-visual, a-priori, and image processing information in the model.

A. Adding information to the model

Here we discuss the information contained in the variance of φ_{12}^{12} and in the pseudo-observations d, h, and in σ_d^2 and σ_h^2.

To investigate the components responsible for the variance of the derived variates φ_{12}^{12}, we propagate the influence of the constituting variates in eq. (1), i.e.,

$$\sigma^2_{\varphi_{12}^{12}} = \left[1, -1, -1, 1\right]\begin{bmatrix}\sigma^2_{\phi_1^1} & & & sym \\ \sigma_{\phi_1^1,\phi_1^2} & \sigma^2_{\phi_1^2} & & \\ \sigma_{\phi_1^1,\phi_2^1} & 0 & \sigma^2_{\phi_2^1} & \\ 0 & \sigma_{\phi_1^2,\phi_2^2} & \sigma_{\phi_2^1,\phi_2^2} & \sigma^2_{\phi_2^2}\end{bmatrix}\begin{bmatrix}1 \\ -1 \\ -1 \\ 1\end{bmatrix} \tag{5}$$

$$= \sum_{i=1}^{2}\sum_{j=1}^{2}\sigma^2_{\phi_j^i} - 2(\sigma_{\phi_1^1,\phi_1^2} - \sigma_{\phi_1^1,\phi_2^1} - \sigma_{\phi_1^2,\phi_2^2} - \sigma_{\phi_2^1,\phi_2^2}) \tag{6}$$

where

- $\sigma^2_{\phi_j^i}$ is the total variance of a phase observation, consisting of thermal noise, scattering noise and atmospheric and orbital signal. Therefore this is a very large number.
- $\sigma_{\phi_1^1,\phi_1^2}$ is the covariance at one position ($p = 1$) between two times. Here the effect of the coherence plays the dominant role; a large coherence implies a large similarity, hence a large covariance.
- $\sigma_{\phi_1^1,\phi_2^1}$ is the covariance between two positions at the same time $t = 1$. This covariance is mainly dependent on the atmospheric signal and orbital residual signal, and is therefore a function of the distance between the two points.

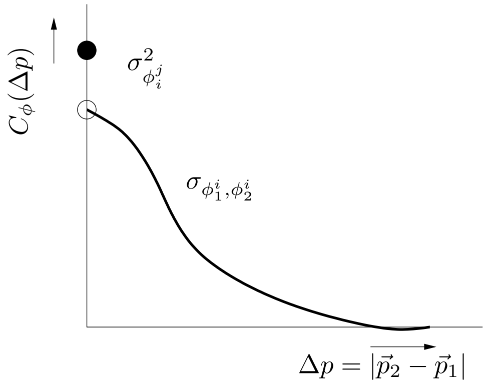

Fig. 1. Sketch of covariance function representing distance dependent covariance at one moment in time. Its shape and magnitude is mainly dependent of the atmospheric conditions and orbit errors. The value for $|\vec{p}_2 - \vec{p}_1| = 0$ corresponds with the total atmospheric variability plus the single point noise (thermal and scattering)

$\sigma_{\phi_1^2,\phi_2^2}$ is the covariance between the two points at time $t = 2$. If no a priori information on the weather situation is available, the same value should be used as for $\sigma_{\phi_1^1,\phi_2^1}$

$\sigma_{\phi_2^1,\phi_2^2}$ is the covariance between the two times for the second point. Again this is a function of the coherence at that point.

Overall, it is clear from eq. (6) that $\sigma^2_{\phi_j^i}$ contains more or less all error sources, but that the variance of the double-difference observation $\sigma^2_{\varphi_{12}^{12}}$ can be strongly reduced if the covariance terms are large as well. Figure 1 demonstrates this. The variance of the double-difference observation between two points would be the value of the covariance function for a zero-distance minus the value at a certain distance Δp. Short distances therefore increase the precision of the estimates.

The (initial) value for the pseudo-observation d for the double-difference deformation between the two points may be hard to predict, although an a priori model for e.g. subsidence might be used. However, if no other information is available, using $E\{d\} = 0$ seems a fair choice. Note that this choice might lead to a biased estimator, but this problem can usually be solved in an iterative sense.

The variance $\sigma_d^2 = f(t_1, t_2, p_1, p_2)$ of the pseudo-observation d includes information available on the deformation, as a function of the times of the acquisitions and the positions of the two points. For example, a spatio-temporal covariance function can be constructed to express the expected amount of deformation over a time interval $\Delta t = t_2 - t_1$ and spatial distance $\Delta p = |\vec{p}_2 - \vec{p}_1|$. Although this might be easier for a subsidence signal than for complicated faulting related to an earthquake, but simple initial models can be improved iteratively using information from the interferogram and the initial results.

For the pseudo-observation for the topographic height a radar-coded a-priori DEM can be used. If no information is available $E\{h\} = 0$ can be used.

The variance $\sigma_h^2 = f(p_1, p_2)$ of the pseudo-observations h

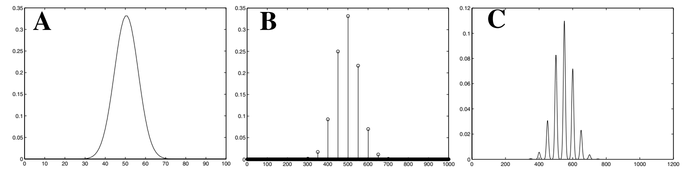

Fig. 2. Sketch of the different probability distributions. **A** is an example of a probability density function for $\hat{D}$ or $\hat{H}$. **B** is the probability mass function for $\check{a}_S$, and **C** is the multi-modal probability density function for $\check{D}$ or $\check{H}$.

expresses the knowledge of the difference in height between two points. This is a function of the distance $\Delta p = |\vec{p}_2 - \vec{p}_1|$ between the two points, as often height differences are less well known over longer distances. If no a priori DEM is available, a fractal dimension and a scale factor could be used to construct the covariance function as a function of distance.

IV. Adjustment and Estimation

Using the model of eqs. (3) and (4) we can now perform a least-squares adjustment procedure using the combined model with integer and real-valued parameters. The procedure to solve the model is divided into three steps [1]. First, the integer nature of the ambiguity a is discarded, and a standard adjustment is performed. The yields real-valued (float) estimates of the parameters and their covariance matrix:

$$\begin{bmatrix} \hat{D} \\ \hat{H} \\ \hat{a} \end{bmatrix} ; \quad \begin{bmatrix} \sigma^2_{\hat{D}} & \sigma_{\hat{H}\hat{D}} & \sigma_{\hat{a}\hat{D}} \\ \sigma_{\hat{D}\hat{H}} & \sigma^2_{\hat{H}} & \sigma_{\hat{a}\hat{H}} \\ \sigma_{\hat{D}\hat{a}} & \sigma_{\hat{H}\hat{a}} & \sigma^2_{\hat{a}} \end{bmatrix} \tag{7}$$

In the second step the float ambiguity estimate $\hat{a}$ is mapped to the corresponding integer (fixed) ambiguity estimate $\check{a}_S = S(\hat{a})$, where S : $\rightarrow$ is the mapping operator. There are several integer estimator that could be chosen to perform this task. In [2] this problem is discussed in relation to radar interferometry. Finally, in the third step the fixed ambiguity estimates are used to correct the float estimates of D and H, as in

$$\begin{bmatrix} \check{D} \\ \check{H} \end{bmatrix} = \begin{bmatrix} \hat{D} \\ \hat{H} \end{bmatrix} - \begin{bmatrix} \sigma_{\hat{a}\hat{D}} \\ \sigma_{\hat{a}\hat{H}} \end{bmatrix} \sigma^2_{\hat{a}} (\hat{a} - \check{a}_S), \tag{8}$$

to retrieve their final 'fixed' estimates,

V. Quality Assessment

It is our goal to determine the quality of the main parameters of interest D and H, preferably in terms of their probability density functions. The quality of the fixed solutions $\check{D}$ and $\check{H}$ depends on the quality of $\hat{D}$, $\hat{H}$, and $\hat{a}$ and on the quality of the integer estimator $\check{a}_S$. Therefore, we need the probability distribution (probability mass function) of the integer estimator $\check{a}_S$. Since there are several integer estimators, the probability distributions of $\check{a}_S$, and consequently $\check{D}$ and $\check{H}$, will be dependent on the estimator chosen. Figure 2 shows what example probability distributions of the float valued parameters (A), fixed integer ambiguity parameters (B), and the multi-modal distribution of the fixed real-valued parameters (C) look like. It is clear that in this example the likelihood peaks of $\check{D}$ or $\check{H}$

are very close, but the main conclusion remains that in order to have a proper quality description of the estimated parameters it is necessary to include the influence of the stochastic integer ambiguities.

VI. Time series

Extension of the model defined in eqs. (3) and (4) to accommodate either time series of point sets, as in persistent scatterer interferometry, or to accommodate spatial differences in one interferogram is relatively straight forward. By estimation of velocity rates or polynomials the number of parameters can be reducedFrom a practical point of view, adding more observations implies an equal increase in ambiguity parameters, which results in a decreased performance of the algorithm in terms of speed. Therefore, recursive bootstrapping methods need to be applied to use results from the previous fixed solutions to add them to the new ones.

VII. Conclusions

This study addressed the problem of quality description in radar interferometry. It is argued that it is possible to include information that is normally used in qualitative interpretation in the functional and stochastic model of a Gauss-Markov parameter estimation formulation. The stochastic nature of the integer ambiguities is treated similar as other (real-valued) parameters, leading to a multi-modal probability distribution of the parameters of interest. The specific shape of the multi-modal distribution, e.g., the space between the likelihood peaks and their relative likelihood magnitudes is a measure that could be used to express the external reliability of the estimated parameters.

References

[1] P. J. G. Teunissen, "The least-squares ambiguity decorellation adjustment: a method for fast GPS integer ambiguity estimation," *Journal of Geodesy*, no. 70, pp. 65–82, 1995.

[2] B. M. Kampes and R. F. Hanssen, "Ambiguity resolution for permanent scatterer interferometry," *IEEE Transactions on Geoscience and Remote Sensing*, vol. accepted for publication, 2004.