

Unit - Wellenhof 97

The subsequent paragraphs explain some out of the many OTF techniques that can be used. Examples are: the ambiguity function method, the least squares ambiguity search, the fast ambiguity resolution approach, the fast ambiguity search filter, least squares ambiguity decorrelation adjustment method, and ambiguity determination with special constraints.

Numerous approaches may be found in recent publications, e.g., the fast ambiguity resolution using an integer nonlinear programming method, see Wei and Schwarz (1995b); a maximum likelihood method based on undifferenced phases, see Knight (1994); the fitting of individual epoch residuals for potential ambiguity candidates to low-order polynomials, see Borge and Forssell (1994). Additional methods may be found in the review papers by Chen and Lachapelle (1994), Hatch (1994), Hein (1995).

### Ambiguity function method

Counselman and Gourevitch (1981) proposed the principle of the ambiguity function, Remondi (1984, 1990a) and Mader (1990) further investigated this method. The concept should become clear from the following description. Assume the model (8.34) for the single-difference phase represented by

$$\Phi_{AB}^j(t) = \frac{1}{\lambda} \varrho_{AB}^j(t) + N_{AB}^j - f \delta_{AB}(t) \quad (9.48)$$

*phase observation, want we clear each ambiguity*

for the points  $A$  and  $B$ , and the satellite  $j$ . If point  $A$  is assumed known and  $B$  is a selected candidate from the gridded cube, then the term  $\varrho_{AB}^j(t)$  is known and may be shifted to the left side of the equation:

$$\Phi_{AB}^j(t) - \frac{1}{\lambda} \varrho_{AB}^j(t) = N_{AB}^j - f \delta_{AB}(t) \quad (9.49)$$

*ambiguities*

The key is to circumvent the ambiguities  $N_{AB}^j$ . A special effect occurs if the term  $2\pi N_{AB}^j$  is used as the argument of a cosine or sine function because  $N_{AB}^j$  is an integer. Therefore, the whole expression (9.49) is multiplied by  $2\pi$  and placed into the complex plane by raising both the left and right side to the power of  $e^i$  where  $i = \sqrt{-1}$  is the imaginary unit. In detail,

$$e^{i\{2\pi \Phi_{AB}^j(t) - \frac{2\pi}{\lambda} \varrho_{AB}^j(t)\}} = e^{i\{2\pi N_{AB}^j - 2\pi f \delta_{AB}(t)\}} \quad (9.50)$$

which may be written as

$$e^{i\{2\pi \Phi_{AB}^j(t) - \frac{2\pi}{\lambda} \varrho_{AB}^j(t)\}} = e^{i 2\pi N_{AB}^j} e^{-i 2\pi f \delta_{AB}(t)} \quad (9.51)$$

It is illustrative to consider this situation in the complex plane, cf. Fig. 9.3. Note the equivalence

$$e^{i\alpha} = \cos \alpha + i \sin \alpha \quad (9.52)$$

Ambiguity function: search of solution space problems.

- labor intensive
- local maxima
- dependent of discretization level
- the quality description is lacking

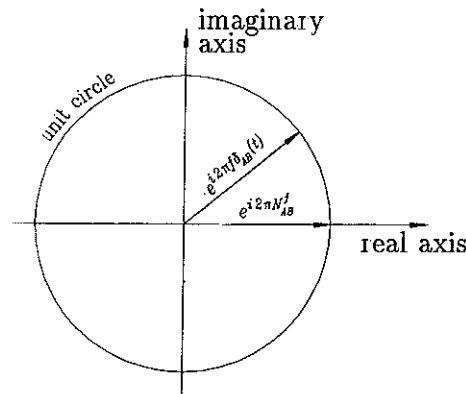


Fig. 9.3. Vector representation in the complex plane

which may be represented as a unit vector with the components  $\cos \alpha$  and  $\sin \alpha$  if a real axis and an imaginary axis are used. Therefore,

$$e^{i2\pi N_{AB}^j} = \cos(2\pi N_{AB}^j) + i \sin(2\pi N_{AB}^j) = 1 + i \cdot 0 \quad (9.53)$$

results because of the integer nature of  $N_{AB}^j$ . Hence, for one epoch and one satellite

$$e^{i\{2\pi \Phi_{AB}^j(t) - \frac{2\pi}{\lambda} \varrho_{AB}^j(t)\}} = e^{-i2\pi f \delta_{AB}(t)} \quad (9.54)$$

remains. Considering  $n_j$  satellites and forming the sum over these satellites for the epoch  $t$  leads to

$$\sum_{j=1}^{n_j} e^{i\{2\pi \Phi_{AB}^j(t) - \frac{2\pi}{\lambda} \varrho_{AB}^j(t)\}} = n_j e^{-i2\pi f \delta_{AB}(t)} \quad (9.55)$$

*did our vector sum*  
*all*  
*satellites*

Considering more than one epoch, the fact that the clock error  $\delta_{AB}(t)$  varies with time must be taken into account. A glance at Fig. 9.3 should recall that  $e^{-i2\pi f \delta_{AB}(t)}$  is a unit vector. Thus, when  $\|e^{-i2\pi f \delta_{AB}(t)}\| = 1$  is applied to (9.55), the relation

$$\left\| \sum_{j=1}^{n_j} e^{i\{2\pi \Phi_{AB}^j(t) - \frac{2\pi}{\lambda} \varrho_{AB}^j(t)\}} \right\| = n_j \cdot 1 \quad (9.56)$$

*norm*

is obtained where the clock error has now vanished.

Take for example four satellites and an error free situation (i.e., neither measurement errors nor model errors, and correct coordinates for the points  $A$  and  $B$ ). In this case, the evaluation of the left side of (9.56) should yield 4 where  $\Phi_{AB}^j(t)$  are the single-differences of measured phases and  $\varrho_{AB}^j(t)$

can be calculated from the known points and satellite positions. However, if point  $B$  was chosen incorrectly then the result must be less than 4. In reality, this maximum can probably never be achieved precisely because of measurement errors and incomplete modeling. Thus, the task is restricted to obtaining the maximum of (9.56) by varying  $B$ .

With highly stable receiver clocks and close epoch spacing it is theoretically possible to include more than one epoch within the absolute value. Using  $n_t$  epochs, the contribution of all epochs may be summed up by

$$\sum_{t=1}^{n_t} \left\| \sum_{j=1}^{n_j} e^{i\{2\pi \Phi_{AB}^j(t) - \frac{2\pi}{\lambda} \rho_{AB}^j(t)\}} \right\| = n_t n_j \quad (9.57)$$

where for simplicity the same number of satellites at all epochs is assumed. Following Remondi (1984, 1990a), the left side of (9.57), i.e., the double sum, is denoted as an ambiguity function. Analogous to the case with one epoch, the maximum of the ambiguity function must be found. In general it will, as before, be less than the theoretical value  $n_t n_j$ .

The ambiguity function procedure is simple. Assume an approximate solution for point  $B$ , e.g., by triple-differences. Then, place this solution into the center of a cube, cf. Fig. 9.4, and partition the cube into grid points. Each grid point is a candidate for the final solution, and the ambiguity function (9.57) is calculated for all single-differences. The grid point yielding the maximum ambiguity function value, which should theoretically be equal to the total number of single-differences (i.e.,  $n_t n_j$ ), is the desired solution. Having found this solution, the ambiguities could be computed using double-differences. Also, an adjustment using double-differences might be performed to verify the position of  $B$  and the ambiguities. The computation of point  $B$  with fixed ambiguities is the final step.

It is worth noting that the ambiguity function method is completely insensitive to cycle slips. The reason can easily be seen from Eq. (9.53). Even if the ambiguity changes by an arbitrary integer amount  $\Delta N_{AB}^j$ , then  $e^{i2\pi(N_{AB}^j + \Delta N_{AB}^j)}$  is still a unit vector and the subsequent equations, therefore,

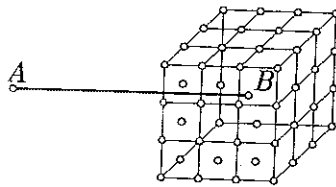


Fig. 9.4. Search technique

remain unchanged. Other methods require that cycle slips be repaired before computing the ambiguity.

Remondi (1984) shows detailed examples of how to speed up the procedure, how to choose the density of the grid points within the cube, and how to find the correct maximum if there are many relative maxima for the ambiguity function. These considerations are significant, since the computational burden could, otherwise, become overwhelming. For illustrative purposes, assume a  $6\text{ m} \times 6\text{ m} \times 6\text{ m}$  cube with a one centimeter grid. Then  $(601)^3 \approx 2.17 \cdot 10^8$  possible solutions must be checked with the ambiguity function (9.57).

#### *Least squares ambiguity search technique*

The method described here is proposed in Hatch (1989) and investigated in further details in Hatch (1990, 1991). The least squares ambiguity technique requires an approximate solution for the position (due to the linearization of the observation equation) which may be obtained from a code range solution. The search area may be established by surrounding the approximate position by a  $3\sigma$  region. One of the basic principles of the approach is the separation of the satellites into a primary and a secondary group. The primary group consists of four satellites. Based on these four satellites, which should have a good PDOP, the possible ambiguity sets are determined. The remaining secondary satellites are used to eliminate candidates of the possible ambiguity sets.

The set of potential solutions may be found in the following way. Assume the simplified double-difference model (9.17). If the ambiguities are moved to the left side as if they were known, the model reads  $\lambda \Phi - N = \varrho$  where all indices have been omitted. For four satellites, three equations of this type may be set up. The three unknown station coordinates contained in the right side of the equation may be solved by linearizing  $\varrho$  and inverting the  $3 \times 3$  design matrix. Specifying and varying the three ambiguities on the left side gives new position solutions where the inverted design matrix remains unchanged. Depending on the variation of the three ambiguities, the set of potential solutions is obtained. Note that Hatch (1990) does not use double-differences but undifferenced phases to avoid any biasing.

From the set of potential solutions, incorrect solutions are removed by taking into account the information of the secondary group of satellites. Sequential least squares adjustment would be appropriately used for this task. Finally, the sum of the squared residuals may be taken as criterion for the quality indicator of the solution. Ideally, only the true set of ambiguities should remain. If this is not the case, then, as described previously, the

solution with the smallest sum of squared residuals should be chosen (after comparing it with the second smallest sum).

*Fast ambiguity resolution approach*

The development of the fast ambiguity resolution approach (FARA) is given in Frei and Beutler (1989), Frei (1991), and summarized in Frei and Schuberl (1992). Following the latter publication, the main characteristics are (1) to use statistical information from the initial adjustment to select the search range, (2) to use information of the variance-covariance matrix to reject ambiguity sets that are not acceptable from the statistical point of view, and (3) to apply statistical hypothesis testing to select the correct set of integer ambiguities.

Following Erickson (1992a), the FARA algorithm may be partitioned into four steps: computing the float carrier phase solution, choosing ambiguity sets to be tested, computing a fixed solution for each ambiguity set, and statistically testing the fixed solution with the smallest variance.

In the first step, real values for double-difference ambiguities are estimated based on carrier phase measurements and calculated by an adjustment procedure which also computes the cofactor matrix of the unknown parameters and the a posteriori variance of unit weight (a posteriori variance factor). Thus, the variance-covariance matrix of the unknown parameters and the standard deviations of the ambiguities may also be computed.

In the second step, the criteria for the ambiguity ranges to be investigated are based on confidence intervals of the real values of the ambiguities. Therefore, the quality of the initial solution of the first step affects the possible ambiguity ranges. In more detail, if  $\sigma_N$  represents the standard deviation of the ambiguity  $N$ , then  $\pm k \sigma_N$  is the search range for this ambiguity where  $k$  is derived statistically from Student's  $t$ -distribution. This is the first criterion for selecting possible ambiguity sets.

A second criterion is the use of the correlation of the ambiguities. Assuming the double-difference ambiguities  $N_i$  and  $N_j$  and the difference

$$N_{ij} = N_j - N_i, \quad (9.58)$$

the standard deviation follows from the error propagation law as

$$\sigma_{N_{ij}} = \sqrt{\sigma_{N_i}^2 - 2\sigma_{N_i N_j} + \sigma_{N_j}^2} \quad (9.59)$$

where  $\sigma_{N_i}^2$ ,  $\sigma_{N_i N_j}$ , and  $\sigma_{N_j}^2$  are contained in the variance-covariance matrix of the parameters. The search range for the ambiguity difference  $N_{ij}$  is  $k_{ij} \sigma_{N_{ij}}$  where  $k_{ij}$  is analogous to the search range for individual double-difference ambiguities. This criterion significantly reduces the number of possible integer sets. An even more impressive reduction is achieved if dual frequency

phase measurements are available. Very illustrative figures demonstrating this reduction are given in Frei and Schubernigg (1992).

In the third step, least squares adjustments with fixed ambiguities are performed for each statistically accepted ambiguity set yielding adjusted baseline components and a posteriori variance factors.

In the fourth and final step, the solution with the smallest a posteriori variance is further investigated. The baseline components of this solution are compared with the float solution. If the solution is compatible, it is accepted. As shown in Erickson (1992a), the compatibility may be checked by a  $\chi^2$ -distribution which tests the compatibility of the a posteriori variance with the a priori variance. Furthermore, another test may be applied to ensure that the second smallest variance is sufficiently less likely than the smallest variance. Note, however, that these two variances are not independent, see Teunissen (1996), Sect. 8.2.3.

As seen from the algorithm, FARA only needs data for double-difference phases; thus, in principle, neither code data nor dual frequency data are required; however, these data will increase the number of possible ambiguity sets dramatically (see the second step of the algorithm).

Euler et al. (1990) present a very efficient and rapid search technique, similar to FARA based on the a posteriori variance (resulting from the sum of the squared residual errors). First, an integer set of ambiguities is introduced in the adjustment computation as constraints leading to an initial solution and the corresponding a posteriori variance. The influence of other ambiguity sets on the initial solution and the a posteriori variance is then determined without recomputing the whole adjustment. This influence may be calculated by some simple matrix and vector operations where only a reduced matrix with the dimension of the constraint ambiguities must be inverted. Following Landau and Euler (1992), the computation time for the matrix inversion may be optimized when the Cholesky factorization method is applied which decomposes a symmetric matrix into a product of a lower and an upper triangle matrix. The impact of a changed ambiguity set on the sum of the squared residuals may be reduced by the Cholesky factorization to the computation of an inner product of two vectors. Furthermore, not even the full inner product must be computed in all cases. Based on a threshold, the computation of the inner product for some integer ambiguity sets may be interrupted and the corresponding ambiguity set rejected.

The performance of this method is demonstrated in Landau and Euler (1992) by imposing examples. Assuming six satellites and therefore five double-difference ambiguities with a  $\pm 10$  cycle uncertainty each, the total number of possible combinations is 3.2 millions. Using a 486 PC, the computation by the Cholesky factorization took 49.1 seconds. Optimizing the

Cholesky factorization by introducing the above mentioned threshold for the inner product, the computation time reduces to 0.2 seconds. For a larger search window of  $\pm 50$  cycles, the corresponding computations amount to 1.5 days for the Cholesky factorization and 3 seconds for the optimized method. The method may be extended to dual frequency data. The appropriate formulas are given in Landau and Euler (1992).

The search techniques described so far performed the search in the ambiguity domain. An alternate technique substitutes the position as known and solves for the ambiguities as unknowns. This could be performed in the following way. Eliminate the ambiguities by forming triple-differences and obtain a first estimate for the position and its standard deviation  $\sigma$  by an adjustment. Now center the approximate position within a cube of dimension  $\pm 3\sigma$  in each coordinate direction and partition the cube into a regular grid. The cube, thus, contains a matrix of points where the center point is the triple-difference solution, see Fig. 9.4. Each of these grid points is considered a candidate for the correct solution. Consequently, one by one, each candidate position is substituted into the observation equation. Then the adjustment (holding the trial position fixed) is performed and the ambiguities are computed. When all points within the cube have been considered, select the solution where the estimated real values of the ambiguities appear as close as possible to integer values. Now, fix the ambiguities to these integer values and compute (holding the ambiguities fixed) the final position which will, in general, be slightly different from the corresponding grid point of the cube.

#### *Fast ambiguity search filter*

Following Chen (1994) and Chen and Lachapelle (1994), the fast ambiguity search filtering algorithm (FASF) is comprised of basically three components: (1) a Kalman filter is applied to predict a state vector which is treated as observable, (2) the search of the ambiguities is performed at every epoch until they are fixed, and (3) the search ranges for the ambiguities are computed recursively and are related to each other.

By applying the Kalman filter, information from the initial epoch to the current epoch is taken into account. The state vector of the Kalman filter also contains the ambiguities which are estimated as real numbers if they cannot be fixed. After fixing the ambiguities, the state vector is modified accordingly. The state vector of the Kalman filter is considered an observable and establishes, along with the regular observables (i.e., double-difference phase equations), the design matrix.

The recursively determined search ranges are based on the a priori geometric information and the effect of other (preliminarily) fixed ambiguities.

As an example, take the case of four double-difference ambiguities. The first ambiguity is computed without fixing any other ambiguity. The search range for the second ambiguity is computed where the first ambiguity is introduced as a known integer quantity (although it may even be the wrong integer number), the search range for the third ambiguity is computed where the first and the second ambiguity are introduced as known integer quantities, and the procedure is continued for the fourth ambiguity. According to Chen and Lachapelle (1994), this concept is denoted as recursive computation of the search range. This recursive computation is similar to nested loops used in computer programs. Referring to the example of the four ambiguities, four nested loops are required where the loop of the first ambiguity is the outermost and the loop of the fourth ambiguity is the innermost loop. It is important to note that the ranges of the loops (apart from the outermost loop) are computed based on the values of the corresponding outer loop indexes. Thus, e.g., the search range of the second loop is determined by using the ambiguity value corresponding to the first (and outermost) loop index.

To avoid very large search ranges, a computational threshold is used. Ambiguities which cross this threshold are not fixed but computed as real numbers. Thus, an attempt to fix the ambiguities is only made if the number of potential ambiguity sets is below this threshold. Under normal circumstances, the number of potential ambiguity sets should decrease with accumulating observations. Ideally, there should finally remain a single potential ambiguity set. In practice, however, this will usually not be the case so that, conventionally, a ratio test of the sum of the squared residuals between the minimum and the second best minimum is calculated. If this ratio fulfills a specified criterion number, the minimum solution is considered to yield the true set of ambiguities.

Once the ambiguities are fixed properly, they are removed from the state vector of the Kalman filter, i.e., from the estimation. Accordingly, the corresponding observation equation is rearranged.

The ranges of the loops for the ambiguities, i.e., the uncertainties, are calculated by using a least squares approach with parameter elimination. First, the parameters representing the station coordinates are eliminated from the normal equations so that the ambiguities are the only remaining parameters of the model based on double-differences. Furthermore, according to the previous discussion on loops associated with the ambiguities, the ambiguities of the outer loops are constrained as integers (even if they may be wrong values). Returning to the example of the four ambiguities, if the range of the third ambiguity is to be determined, the first and the second ambiguity are assumed to be known and introduced as constraints (which is equivalent to removing them from the estimation vector). In fact, this may be done very



efficiently as shown in Chen and Lachapelle (1994) where only single rows and columns of the adjustment matrices must be taken into account. As result of this parameter elimination, a float estimation of the corresponding ambiguity and its variance are finally obtained. Multiplying the variance by a scale factor and subtracting and adding this result with respect to the float solution yields the search range for this specific ambiguity.

Note that if the uncertainty ranges are not calculated correctly, the true ambiguity set will not be found.

*Least squares ambiguity decorrelation adjustment method*

Teunissen (1993) proposed the idea and further developed the least squares ambiguity decorrelation adjustment (Lambda) method. A fairly detailed description of Teunissen's method is (slightly modified) given here. At present, this method is both theoretically and practically at the top level among the ambiguity determination methods.

The conventional formulation of the adjustment by parameters is

$$\underline{n}^T \underline{P} \underline{n} = \text{minimum} \quad (9.60)$$

where  $\underline{n}$  is the vector of residuals and  $\underline{P}$  is the weight matrix. This formulation implies that the weighted sum of squared residuals is minimized. As shown in Sect. 9.3.1, the weight matrix equals the inverse of the cofactor matrix  $\underline{Q}$  of observations. Consequently,

$$\underline{n}^T \underline{Q}^{-1} \underline{n} = \text{minimum} \quad (9.61)$$

is an equivalent relation. One remark is appropriate here. Imprecisely, the cofactor matrix is frequently also denoted as covariance matrix. Theoretically, the difference between the two matrices is the a priori variance of unit weight acting as a scale factor which may be arbitrarily chosen. Thus, if this scale factor is set equal to one, the cofactor matrix equals the covariance matrix.

Applying least squares adjustment for, e.g., relative positioning based on double-difference phase observations, the unknowns being determined are coordinate increments for the unknown station and double-difference ambiguities. The values obtained from the adjustment procedures are in the sense of this minimum principle the most likely ones. However, the double-difference ambiguities are obtained as real values (often denoted as float ambiguities) but should be integer values. The main objective is, thus, to obtain integer ambiguities which are the most likely ones. Denoting the vector of adjusted float ambiguities by  $\hat{\underline{N}}$  and the vector of the corresponding integer ambiguities by  $\underline{N}$ , the difference between the two vectors may be

regarded as residuals of ambiguities. Consequently, it makes sense to minimize these residuals again by the same principle, i.e., the weighted sum of squared residuals. Explicitly,

$$(\hat{N} - N)^T \underline{Q}_{\hat{N}}^{-1} (\hat{N} - N) = \text{minimum} \quad (9.62)$$

is obtained where  $\underline{Q}_{\hat{N}}$  is the cofactor or covariance matrix (see discussion above!) of the adjusted float ambiguities. Teunissen et al. (1995) introduce the short-hand notation

$$\chi^2(N) = (\hat{N} - N)^T \underline{Q}_{\hat{N}}^{-1} (\hat{N} - N) = \text{minimum} \quad (9.63)$$

and denote the solution of this problem the integer least squares estimate of the ambiguities. Certainly, an approach different from the usual least squares adjustment calculation must be chosen to account for the integer nature of the still unknown ambiguities  $N$ .

The following simple example demonstrates the solution principle. Considering two ambiguities and assuming  $\underline{Q}_{\hat{N}}$  as unit matrix

$$\underline{Q}_{\hat{N}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (9.64)$$

equation (9.63) reduces to

$$\chi^2(N) = (\hat{N}_1 - N_1)^2 + (\hat{N}_2 - N_2)^2 \quad (9.65)$$

which is the sum of independently squared ambiguity residuals. The function  $\chi^2(N)$  becomes a minimum, if each squared residual  $(\hat{N}_i - N_i)^2$ ;  $i = 1, 2$  is minimized individually. Obviously, the minimum is achieved if the  $N_i$  are chosen as those integer values being nearest to the real values. In other words, rounding the real value ambiguities to their nearest integer values yields the desired minimum for  $\chi^2(N)$ .

Since  $\underline{Q}_{\hat{N}}$  was assumed as unit matrix, the resulting  $N_1$  and  $N_2$  are fully decorrelated which is also evident from Eq. (9.65). Geometrically, if two coordinate axes are associated with  $N_1$  and  $N_2$ , this equation represents a circle centered around the ambiguities  $\hat{N}$  and with radius  $\chi(N)$ . This circle is regarded as ambiguity search space. Mathematically, the two integer ambiguities are contained in the two-dimensional integer space.

It is more general to assume  $\underline{Q}_{\hat{N}}$  as diagonal matrix. Using again two ambiguities, the matrix

$$\underline{Q}_{\hat{N}} = \begin{bmatrix} q_{\hat{N}_1 \hat{N}_1} & 0 \\ 0 & q_{\hat{N}_2 \hat{N}_2} \end{bmatrix} \quad (9.66)$$

yields the result

$$\chi^2(\underline{N}) = \frac{(\hat{N}_1 - N_1)^2}{q_{\hat{N}_1 \hat{N}_1}} + \frac{(\hat{N}_2 - N_2)^2}{q_{\hat{N}_2 \hat{N}_2}} \quad (9.67)$$

which leads to the same conclusion, namely the minimum of  $\chi^2(\underline{N})$  is obtained if the real value ambiguities are rounded to their nearest integer values.

Since  $\underline{Q}_{\hat{N}}$  was assumed as diagonal matrix, the resulting  $N_1$  and  $N_2$  are still fully decorrelated which is also evident from Eq. (9.67). Geometrically, if two coordinate axes are associated with  $N_1$  and  $N_2$ , this equation represents an ellipse centered around the ambiguities  $\hat{N}$  and with the semiaxes

$$\begin{aligned} a &= \chi(\underline{N}) \sqrt{q_{\hat{N}_1 \hat{N}_1}} \\ b &= \chi(\underline{N}) \sqrt{q_{\hat{N}_2 \hat{N}_2}} \end{aligned} \quad (9.68)$$

where  $\chi(\underline{N})$  acts as a scale factor. The axes of the ellipse are parallel to the direction of the coordinate axes. This ellipse is regarded as an ambiguity search space. Mathematically, the two integer ambiguities are contained in the two-dimensional integer space.

In reality,  $\underline{Q}_{\hat{N}}$  will be a fully occupied symmetric matrix. Restricting again to two ambiguities, this matrix reads

$$\underline{Q}_{\hat{N}} = \begin{bmatrix} q_{\hat{N}_1 \hat{N}_1} & q_{\hat{N}_1 \hat{N}_2} \\ q_{\hat{N}_1 \hat{N}_2} & q_{\hat{N}_2 \hat{N}_2} \end{bmatrix} \quad (9.69)$$

Substituting this matrix into the general minimum requirement (9.63) yields

$$\chi^2(\underline{N}) = \begin{bmatrix} \hat{N}_1 - N_1 \\ \hat{N}_2 - N_2 \end{bmatrix}^T \begin{bmatrix} q_{\hat{N}_1 \hat{N}_1} & q_{\hat{N}_1 \hat{N}_2} \\ q_{\hat{N}_1 \hat{N}_2} & q_{\hat{N}_2 \hat{N}_2} \end{bmatrix}^{-1} \begin{bmatrix} \hat{N}_1 - N_1 \\ \hat{N}_2 - N_2 \end{bmatrix} \quad (9.70)$$

and, after carrying out the matrix-vector multiplications,

$$\bar{\chi}^2(\underline{N}) = \frac{(\hat{N}_1 - N_1)^2}{q_{\hat{N}_1 \hat{N}_1}} + \frac{(\hat{N}_2 - N_2)^2}{q_{\hat{N}_2 \hat{N}_2}} - \frac{(\hat{N}_1 - N_1)(\hat{N}_2 - N_2)q_{\hat{N}_1 \hat{N}_2}}{q_{\hat{N}_1 \hat{N}_1} q_{\hat{N}_2 \hat{N}_2}} \quad (9.71)$$

is obtained where

$$\bar{\chi}^2(\underline{N}) = \chi^2(\underline{N}) \left( 1 - \frac{q_{\hat{N}_1 \hat{N}_2}^2}{q_{\hat{N}_1 \hat{N}_1} q_{\hat{N}_2 \hat{N}_2}} \right) \quad (9.72)$$

Equation (9.71) still represents ellipses but their axes are rotated with respect to the coordinate system associated with  $N_1$  and  $N_2$  which implies a correlation of the two ambiguities. Due to this correlation, expressed by the third term on the right side of (9.71), it is more complicated to find the minimum for  $\bar{\chi}^2(\underline{N})$ . In other words, the rounding to the nearest integer principle no longer works. To return to this convenient feature, the idea is to apply a transformation that decorrelates the ambiguities which means that the transformed covariance matrix of the ambiguities becomes a diagonal matrix.

Finding a transformation that produces a diagonal matrix for  $\underline{Q}_{\hat{N}}$  seems to be trivial since an eigenvalue decomposition yields a diagonal matrix as output. Explicitly, each symmetric matrix

$$\underline{Q} = \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} \quad (9.73)$$

can be transformed into the diagonal matrix

$$\underline{Q}' = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad (9.74)$$

The eigenvalues are defined by

$$\begin{aligned} \lambda_1 &= \frac{1}{2}(q_{11} + q_{22} + w) \\ \lambda_2 &= \frac{1}{2}(q_{11} + q_{22} - w) \end{aligned} \quad (9.75)$$

with the auxiliary quantity

$$w = \sqrt{(q_{11} - q_{22})^2 + 4q_{12}^2} \quad (9.76)$$

The two eigenvectors are orthogonal to each other and are defined by the rotation angle  $\varphi$  which can be calculated by

$$\tan 2\varphi = \frac{2q_{12}}{q_{11} - q_{22}} \quad (9.77)$$

The only problem is that the integer ambiguities  $\underline{N}$  must also be transformed and must preserve their integer nature. Thus, an ordinary eigenvalue decomposition will not work.

Generally, the task may be formulated in the following way. The ambiguities  $\underline{N}$  and  $\hat{\underline{N}}$  are reparameterized by matrix  $\underline{Z}$ . Note that Teunissen uses the transposed matrix  $\underline{Z}^T$ , but the principle remains the same. Hence,

$$\begin{aligned} \underline{N}' &= \underline{Z} \underline{N} \\ \hat{\underline{N}}' &= \underline{Z} \hat{\underline{N}} \\ \underline{Q}_{\hat{N}'} &= \underline{Z} \underline{Q}_{\hat{N}} \underline{Z}^T \end{aligned} \quad (9.78)$$

where the transformation of the cofactor matrix is obtained by applying the error propagation law. The ambiguities  $\underline{N}'$  obtained after transformation must remain integer values. That restricts the matrix  $\underline{Z}$  to a specific class of transformations, see Teunissen (1995), where three conditions must be fulfilled. Following Teunissen (1994), these conditions are: (1) the elements of the transformation matrix  $\underline{Z}$  must be integer values, (2) the transformation must be volume preserving, and (3) the transformation must reduce the product of all ambiguity variances.

Note that the inverse of the transformation matrix  $\underline{Z}$  must also consist of integer values only, because upon a retransformation of the (determined) integer ambiguities  $\underline{N}'$ , the integer nature of the ambiguities must be kept.

For the two-dimensional example shown, volume preserving reduces to area preserving of the ellipse represented by the two-dimensional cofactor (covariance) matrix.

If the three conditions are fulfilled, the transformed integer ambiguities are again integer values and the cofactor (covariance) matrix of the transformed ambiguities is more diagonal than the cofactor (covariance) matrix of the original ambiguities, cf. Teunissen (1994).

The Gauss transformation is one of the possible candidates. In two dimensions, the Gauss transformation is given as

$$\underline{Z} = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} \quad (9.79)$$

where  $\alpha$  may be chosen arbitrarily. Applying this transformation to  $\hat{N}$  yields

$$\begin{bmatrix} \hat{N}'_1 \\ \hat{N}'_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} \begin{bmatrix} \hat{N}_1 \\ \hat{N}_2 \end{bmatrix} \quad (9.80)$$

giving

$$\begin{aligned} \hat{N}'_1 &= \hat{N}_1 \\ \hat{N}'_2 &= \alpha \hat{N}_1 + \hat{N}_2 \end{aligned} \quad (9.81)$$

and

$$\underline{Q}_{\hat{N}'} = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} \begin{bmatrix} q_{\hat{N}_1 \hat{N}_1} & q_{\hat{N}_1 \hat{N}_2} \\ q_{\hat{N}_1 \hat{N}_2} & q_{\hat{N}_2 \hat{N}_2} \end{bmatrix} \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \quad (9.82)$$

Denoting the elements of the symmetric  $\underline{Q}_{\hat{N}'}$  as

$$\underline{Q}_{\hat{N}'} = \begin{bmatrix} q_{\hat{N}'_1 \hat{N}'_1} & q_{\hat{N}'_1 \hat{N}'_2} \\ q_{\hat{N}'_1 \hat{N}'_2} & q_{\hat{N}'_2 \hat{N}'_2} \end{bmatrix} \quad (9.83)$$

results in

$$\begin{aligned} q_{\hat{N}'_1 \hat{N}'_1} &= q_{\hat{N}_1 \hat{N}_1} \\ q_{\hat{N}'_1 \hat{N}'_2} &= \alpha q_{\hat{N}_1 \hat{N}_1} + q_{\hat{N}_1 \hat{N}_2} \\ q_{\hat{N}'_2 \hat{N}'_2} &= \alpha^2 q_{\hat{N}_1 \hat{N}_1} + 2\alpha q_{\hat{N}_1 \hat{N}_2} + q_{\hat{N}_2 \hat{N}_2} \end{aligned} \quad (9.84)$$

To determine  $\alpha$ , two arguments may be used which lead to the same result. First,  $\underline{Q}_{\hat{N}'}$  should become a diagonal matrix, i.e.,  $q_{\hat{N}'_1 \hat{N}'_2}$  must become zero. Hence, from the second equation in (9.84) follows  $\alpha = -q_{\hat{N}_1 \hat{N}_2} / q_{\hat{N}_1 \hat{N}_1}$ . Second, find a value for  $\alpha$  that minimizes  $q_{\hat{N}'_2 \hat{N}'_2}$ . This need arises from the third condition of the transformation, namely, the product of all ambiguity variances must be reduced. Since the variance of the first ambiguity remains unchanged due to  $q_{\hat{N}'_1 \hat{N}'_1} = q_{\hat{N}_1 \hat{N}_1}$ , the reduction of the ambiguity variances is maximized if  $q_{\hat{N}'_2 \hat{N}'_2}$  is minimized. Thus, differentiating  $q_{\hat{N}'_2 \hat{N}'_2}$  with respect to  $\alpha$  and setting the result equal to zero yields the same expression  $\alpha = -q_{\hat{N}_1 \hat{N}_2} / q_{\hat{N}_1 \hat{N}_1}$  for  $\alpha$  again. Substituting this result into (9.79) gives

$$\underline{Z} = \begin{bmatrix} 1 & 0 \\ -q_{\hat{N}_1 \hat{N}_2} / q_{\hat{N}_1 \hat{N}_1} & 1 \end{bmatrix} \quad (9.85)$$

and accordingly to (9.80) for the ambiguity transformation

$$\begin{bmatrix} \hat{N}'_1 \\ \hat{N}'_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -q_{\hat{N}_1 \hat{N}_2} / q_{\hat{N}_1 \hat{N}_1} & 1 \end{bmatrix} \begin{bmatrix} \hat{N}_1 \\ \hat{N}_2 \end{bmatrix} \quad (9.86)$$

which may be recognized as the conditional least squares estimate for  $\hat{N}'_2$ , see Teunissen (1994). Conditional least squares estimate means that the estimate for  $\hat{N}'_2$  is conditioned on  $\hat{N}'_1$ . The element  $\alpha = -q_{\hat{N}_1 \hat{N}_2} / q_{\hat{N}_1 \hat{N}_1}$  will most likely violate the condition that the elements of the transformation matrix must be integer values. Therefore, an approximate decorrelation is obtained if this element is rounded to its nearest integer value. Symbolically, this will be expressed by  $-\text{INT}[q_{\hat{N}_1 \hat{N}_2} / q_{\hat{N}_1 \hat{N}_1}]$  where the operator INT performs (like an intrinsic function in a computer language) the rounding to the nearest integer. Therefore, the transformation matrix becomes

$$\underline{Z} = \begin{bmatrix} 1 & 0 \\ -\text{INT}[q_{\hat{N}_1 \hat{N}_2} / q_{\hat{N}_1 \hat{N}_1}] & 1 \end{bmatrix} \quad (9.87)$$

and the transformed ambiguities

$$\begin{bmatrix} \hat{N}'_1 \\ \hat{N}'_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\text{INT}[q_{\hat{N}_1 \hat{N}_2} / q_{\hat{N}_1 \hat{N}_1}] & 1 \end{bmatrix} \begin{bmatrix} \hat{N}_1 \\ \hat{N}_2 \end{bmatrix} \quad (9.88)$$

are, as just mentioned, an approximation to fully decorrelated ambiguities.

The role of the two ambiguities may be interchanged. In the transformation above, the ambiguity  $\hat{N}_1$  remains unchanged and  $\hat{N}_2$  is transformed. Analogously,  $\hat{N}_2$  may be kept unchanged and  $\hat{N}_1$  will be transformed. For a better distinction of the two transformations, the subscripts 1 and 2 are introduced. Thus, the transformation under consideration is now expressed by

$$\underline{Z}_1 = \begin{bmatrix} 1 & 0 \\ \alpha_1 & 1 \end{bmatrix} \quad \alpha_1 = -\text{INT}[q_{\hat{N}_1\hat{N}_2}/q_{\hat{N}_1\hat{N}_1}] \quad (9.89)$$

The other Gauss transformation has the form

$$\underline{Z}_2 = \begin{bmatrix} 1 & \alpha_2 \\ 0 & 1 \end{bmatrix} \quad \alpha_2 = -\text{INT}[q_{\hat{N}_1\hat{N}_2}/q_{\hat{N}_2\hat{N}_2}] \quad (9.90)$$

where the computation of  $\alpha_2$  was performed in a similar manner to the previous case. The transformed ambiguities are obtained from

$$\begin{bmatrix} \hat{N}'_1 \\ \hat{N}'_2 \end{bmatrix} = \begin{bmatrix} 1 & -\text{INT}[q_{\hat{N}_1\hat{N}_2}/q_{\hat{N}_2\hat{N}_2}] \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{N}_1 \\ \hat{N}_2 \end{bmatrix} \quad (9.91)$$

For a numerical example, Teunissen (1996), Sect. 8.5.2, assumed that after a least squares adjustment the ambiguities

$$\underline{\hat{N}} = \begin{bmatrix} \hat{N}_1 \\ \hat{N}_2 \end{bmatrix} = \begin{bmatrix} 1.05 \\ 1.30 \end{bmatrix}$$

and

$$\underline{Q}_{\hat{N}} = \begin{bmatrix} q_{\hat{N}_1\hat{N}_1} & q_{\hat{N}_1\hat{N}_2} \\ q_{\hat{N}_1\hat{N}_2} & q_{\hat{N}_2\hat{N}_2} \end{bmatrix} = \begin{bmatrix} 53.4 & 38.4 \\ 38.4 & 28.0 \end{bmatrix}$$

were calculated. Now the transformation is applied to  $\underline{Q}_{\hat{N}}$ . Translating the matrix elements to variances, the ambiguity  $\hat{N}_1$  has a larger variance than  $\hat{N}_2$ . Hence it is preferable first to change  $\hat{N}_1$  and keep  $\hat{N}_2$  unchanged, i.e., to apply a transformation based on  $\underline{Z}_2$ . From (9.90),

$$\alpha_2 = -\text{INT}[q_{\hat{N}_1\hat{N}_2}/q_{\hat{N}_2\hat{N}_2}] = -\text{INT}[38.4/28.0] = -1$$

and

$$\underline{Z}_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

are obtained. The transformation according to (9.78) reads

$$\underline{Q}_{\hat{N}'} = \underline{Z}_2 \underline{Q}_{\hat{N}} \underline{Z}_2^T = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 53.4 & 38.4 \\ 38.4 & 28.0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

and gives

$$\underline{Q}_{\hat{N}'} = \begin{bmatrix} 4.6 & 10.4 \\ 10.4 & 28.0 \end{bmatrix}.$$

The effect of this transformation can be seen best if the ambiguity search space, represented by the standard ellipse (which is centered around the corresponding ambiguities), is considered. The parameters of the standard ellipse follow from (9.73) through (9.77) if  $\underline{Q}$  is replaced by  $\underline{Q}_{\hat{N}}$  and  $\underline{Q}_{\hat{N}'}$ , respectively. The eigenvalues of the matrices equal the squared semiaxes of the ellipse and  $\varphi$  defines the direction of the semimajor axis. Explicitly, the data

$$\begin{aligned} \underline{Q}_{\hat{N}} : \quad a &= 9.0, \quad b = 0.5, \quad \varphi = 35^\circ \\ \underline{Q}_{\hat{N}'} : \quad a &= 5.7, \quad b = 0.8, \quad \varphi = 69^\circ \end{aligned}$$

are obtained. Graphically, the standard ellipses are shown in Fig. 9.5. The standard ellipse for  $\underline{Q}_{\hat{N}}$  is centered around the ambiguities  $\hat{N}$ , i.e., the origin is at  $\hat{N}_1 = 1.05$  and  $\hat{N}_2 = 1.30$ . The standard ellipse for  $\underline{Q}_{\hat{N}'}$  is centered around the ambiguities  $\hat{N}'$ , i.e., the origin follows from  $\hat{N}' = \underline{Z}_2 \hat{N}$  and amounts to  $\hat{N}'_1 = -0.25$  and  $\hat{N}'_2 = 1.30$ .

The figure also indicates search windows with sides parallel to the two axes of the two-dimensional integer search space, i.e., two horizontal and two vertical tangents of the ellipse. The "volumes" of the two ellipses are the same because the transformation is volume preserving, but the shape and the orientation of the ellipse has changed. The distance between the two horizontal tangents has not changed because these two tangents bound the search range for the  $N_2$  ambiguity which remained unaltered by the  $\underline{Z}_2$  transformation, whereas the distance of the two vertical tangents has changed.

Each grid point represents one pair of ambiguities. Under the assumption that each grid point of the search window must be regarded as a possible candidate to be investigated for a reasonable solution, the advantage of the transformed search space becomes obvious.

From comparing the off-diagonal elements of  $\underline{Q}_{\hat{N}}$  and of the transformed  $\underline{Q}_{\hat{N}'}$ , the decrease of correlation is evident.



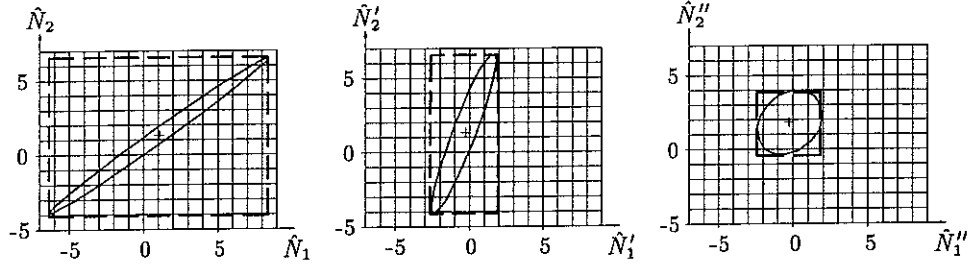


Fig. 9.5. Ambiguity search space for  $\underline{Q}_{\hat{N}}$  (left) and transformed ambiguity search spaces for  $\underline{Q}_{\hat{N}'}$  (middle) and  $\underline{Q}_{\hat{N}''}$  (right)

Another transformation may now be applied to  $\underline{Q}_{\hat{N}'}$ . Since ambiguity  $\hat{N}_2'$  has a larger variance than  $\hat{N}_1'$  it is preferable to change  $\hat{N}_2'$  and keep  $\hat{N}_1'$  unchanged, i.e., to apply a transformation based on  $\underline{Z}_1$ . First, from (9.89)

$$\alpha_1 = -\text{INT}[q_{\hat{N}_1\hat{N}_2}/q_{\hat{N}_1\hat{N}_1}] = -\text{INT}[10.4/4.6] = -2$$

is determined giving

$$\underline{Z}_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

and

$$\underline{Q}_{\hat{N}''} = \underline{Z}_1 \underline{Q}_{\hat{N}'} \underline{Z}_1^T = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 4.6 & 10.4 \\ 10.4 & 28.0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

where the double prime expresses that the transformation is applied on the once transformed matrix. The result is

$$\underline{Q}_{\hat{N}''} = \begin{bmatrix} 4.6 & 1.2 \\ 1.2 & 4.8 \end{bmatrix}$$

The standard ellipse for  $\underline{Q}_{\hat{N}''}$  is given by  $a = 2.4$ ,  $b = 1.9$ ,  $\varphi = 47^\circ$  and is shown in Fig. 9.5. The standard ellipse for  $\underline{Q}_{\hat{N}''}$  is centered around the ambiguities  $\hat{N}''$ , i.e., the origin follows from  $\hat{N}'' = \underline{Z}_1 \hat{N}'$  and amounts to  $\hat{N}_1'' = -0.25$  and  $\hat{N}_2'' = 1.80$ . As far as the search window is concerned, the effect may easily be seen from the much smaller search area (represented by the window) of  $\underline{Q}_{\hat{N}''}$ .

Accordingly, the distance between the two vertical tangents has not changed because these two tangents bound the search range for the  $N_1$  ambiguity which remained unaltered by the  $\underline{Z}_1$  transformation, whereas the distance of the two horizontal tangents has changed.

From comparing the off-diagonal elements of  $\underline{Q}_{\hat{N}'}$  and of the transformed  $\underline{Q}_{\hat{N}''}$ , the decrease of correlation is evident. However, the ambiguities are still not fully decorrelated.

The two transformations may also be combined to a single transformation. Using  $\underline{Q}_{\hat{N}''} = \underline{Z}_1 \underline{Q}_{\hat{N}'} \underline{Z}_1^T$  and substituting  $\underline{Q}_{\hat{N}'} = \underline{Z}_2 \underline{Q}_{\hat{N}} \underline{Z}_2^T$  leads to

$$\underline{Q}_{\hat{N}''} = \underbrace{\underline{Z}_1 \underline{Z}_2}_{\underline{Z}} \underline{Q}_{\hat{N}} \underbrace{\underline{Z}_2^T \underline{Z}_1^T}_{\underline{Z}^T}$$

where

$$\underline{Z} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

so that now the single transformation matrix  $\underline{Z}$  represents the composition of the  $\underline{Z}_2$  and the  $\underline{Z}_1$  transformation.

The extension of the reparameterization of the ambiguity search space to higher dimensions is possible. Teunissen (1996), Sect. 8.5.3, gives the decorrelating ambiguity transformation  $\underline{Z}$  for the three-dimensional case which would apply if double-differences of four satellites are used, and a twelve-dimensional transformation for seven satellites and dual frequency data. Rizos and Han (1995) propose an iterative procedure to generate the decorrelating ambiguity transformation  $\underline{Z}$ . Note that the ambiguity search space becomes an ellipsoid for the three-dimensional example and an  $n$ -dimensional hyperellipsoid for  $n > 3$ .

After the decorrelation of the ambiguities by the  $\underline{Z}$  transformation, the task of actually solving ambiguity estimates remains. The search can be carried out very efficiently by using the sequential conditional adjustment, a standard technique in adjustment theory. Related to the ambiguity estimation, an overview is given in Jonge and Tiberius (1995) and some details are covered in Teunissen (1996), Sect. 8.3.2. The sequential conditional adjustment determines the ambiguities step by step (i.e., sequential) one after the other. For the  $i$ -th ambiguity to be estimated, the previously determined  $i - 1$  ambiguities are fixed (i.e., conditional). The sequential conditional least squares adjustment ambiguities are not correlated. This means that the effect of the  $\underline{Z}$  transformation will not be destroyed.

Some details on the actual discrete search strategy are given in Teunissen (1994), Teunissen et al. (1994), Teunissen (1996), Sects. 8.3.2, 8.5.3.

In summary, Teunissen's Lambda method may be separated into the following steps:

1. A conventional least squares adjustment is carried out to yield the baseline components and float ambiguities.

2. Using the  $\underline{Z}$  transformation, the ambiguity search space is reparameterized to decorrelate the float ambiguities.
3. Using the sequential conditional least squares adjustment together with a discrete search strategy, the integer ambiguities are estimated. By the inverse transformation  $\underline{Z}^{-1}$ , the ambiguities are retransformed to the original ambiguity space where the baseline components are given. Since  $\underline{Z}^{-1}$  consists only of integer elements, the integer nature of the ambiguities is kept.
4. The integer ambiguities are fixed as known quantities and another conventional least squares adjustment to determine the final baseline components is performed.

#### *Ambiguity determination with special constraints*

Several multiple receiver methods for kinematic applications exist. One common procedure of this technique is to place two or more receivers at fixed locations (usually short distances apart) of the moving object. Since the locations of the antennas are fixed, constraints (e.g., the distance between two antennas) may be formulated which can be used to increase the efficiency of the ambiguity resolution. In principle, the gain by using constraints results in a reduction of the potential ambiguity sets. This is illustrated briefly by two examples; namely, attitude determination and aircraft-to-aircraft positioning.

The example of attitude determination in a marine environment is taken from Lu and Cannon (1994) and employs the distances between the antennas on a ship as constraints for the ambiguity resolution. Attitude determination is explained in Sect. 12.2. Here, only the principle of the ambiguity resolution with the constraint of the known distance for a single baseline is described. Referring to the double-difference model (9.17), four satellites yielding three double-differences are considered. Analogously to the procedure for the least squares ambiguity search technique described earlier, the equations are reformulated as  $\lambda \Phi - N = \varrho$  where all indices have been omitted. The specification of  $N$ , the double-difference ambiguities, is accomplished by searching a possible set of ambiguity solutions. The most conservative method to define the search space would be to take the length of the baseline and divide it by the wavelength of the carrier used, i.e., 19 cm in the case of L1. The resulting number of cycles taken once positively and once negatively and considered for each ambiguity defines the search area. Various methods exist to reduce the burden of the search. Jurgens et al. (1991) propose for example a third antenna aligned with the other two antennas

(of the baseline to be determined) and located within less than one carrier wavelength from one of the two antennas. In general, the double-difference ambiguities of the two close-by antennas will vanish so that an approximate azimuth and pitch information can be determined and used to reduce the search space for the remaining longer baseline.

Lu and Cannon (1994) and Lu (1995) reduce the search space by introducing the known distance of the baseline. Referring to the system  $\lambda \Phi - N = \varrho$ , three double-differences are considered and the linearization of  $\varrho$  is performed with respect to the reference station of the baseline. Thus, the linearized system may be written as  $\underline{w} = \underline{A}\underline{x}$  where  $\underline{A}$  is a  $3 \times 3$  design matrix resulting from the linearization,  $\underline{x}$  contains the unknown baseline components (since the linearization was carried out with respect to the known station), and the left side of the equation contains the residual vector  $\underline{w}$  which also comprises the ambiguities. Since  $\underline{x}$  represents the baseline components, the constraint of the length of the baseline, denoted by  $b$ , may be introduced by first forming  $\underline{A}^{-1}\underline{w} = \underline{x}$  and then  $b^2 = \underline{x}^T \underline{x} = \underline{w}^T (\underline{A} \underline{A}^T)^{-1} \underline{w}$ . This system may be further simplified by applying to  $\underline{A} \underline{A}^T$  a Cholesky decomposition which reduces  $\underline{A}$  to a lower triangle matrix. The advantage obtained from this decomposition is that the third ambiguity may be expressed by a quadratic equation containing the other two ambiguities. Thus, introducing search trials for these two other ambiguities yields two solutions for the third ambiguity. Therefore, the constraint significantly reduces the search space. Redundant satellites may be used to further reduce the size of the search space.

The performance of this method can best be seen by means of a simple example. Assuming a  $\pm 15$  cycle uncertainty for the three unknowns would yield (together with the one ambiguity set obtained by rounding the calculated unknowns to their nearest integer values)  $31 \times 31 \times 31 = 29\,791$  possible ambiguity sets, whereas taking into account the constraint as described above reduces the set of possible ambiguities to  $31 \times 31 \times 2 = 1\,922$ .

The second example presented here refers to the introduction of constraints for an aircraft-to-aircraft positioning as proposed in Lachapelle et al. (1994). The situation is shown in Fig. 9.6. Each of the two aircraft is equipped with two receivers. The corresponding distances of the antennas between  $i$  and  $j$  on one aircraft and  $k$  and  $\ell$  on the other aircraft are known and may be introduced as constraints to determine the double-difference ambiguities for each airplane separately, i.e., the double-differences  $N_{ij}$  and  $N_{k\ell}$  for the available satellites (which are not indicated here by appropriate superscripts). These resolved ambiguities  $N_{ij}$  and  $N_{k\ell}$  may now be used to interrelate the two aircraft by constraints. As proposed in Lachapelle et al. (1994), three sets of double-difference ambiguity relations are constrained

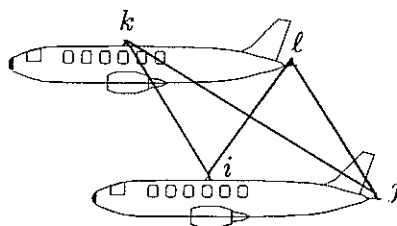


Fig. 9.6. Aircraft-to-aircraft GPS positioning with four receivers

by using, e.g.,  $N_{ij} = N_{jk} - N_{ik}$ ,  $N_{ij} = N_{jl} - N_{il}$ , and  $N_{kl} = N_{li} - N_{ki}$ . Thus, for five satellites there are  $4 \times 3$  double-difference equations of this type which are used to reduce the number of potential ambiguity solutions. Note that these relations are not independent from each other but may still contribute to average out several error sources like carrier phase noise and multipath.

Examples given in Lachapelle et al. (1994) demonstrate that for two aircraft within 1 km typically 4 to 6 minutes of measurements (with a data rate of 1 Hz) are sufficient to obtain a unique solution. The correctness of the ambiguities may roughly be checked by the double-difference phase residuals which must not show a significant drift over time. A drift would be an indication of wrong ambiguities. The rms of the double-difference phase residuals was in the amount of 0.8 cm.

Based on the given data set, several trials were performed by shifting the initial epoch from one trial to the next by 90 seconds. Of these trials, some 50% yielded the same ambiguities. This indicates the correctness of like ambiguity sets; however, this also demonstrates that the reliability of a single solution is not sufficient.

## 9.3 Adjustment, filtering, and smoothing

### 9.3.1 Least squares adjustment

#### *Standard adjustment*

There are numerous adjustment techniques that can be used, but least squares adjustment with parameters is the only one discussed here. It is based on equations where the observations are expressed as a function of unknown parameters. A Taylor series expansion is usually performed in the case of nonlinear functions. This requires approximate values for the parameters. The Taylor series expansion must be truncated after the second term to obtain a linear function with respect to the unknowns. The resulting

linear observation model can be represented in a matrix-vector notation as

$$\underline{\ell} = \underline{A} \underline{x} \quad (9.92)$$

where

$$\begin{array}{ll} \underline{\ell} & \dots \text{ vector of observations} \\ \underline{A} & \dots \text{ design matrix} \\ \underline{x} & \dots \text{ vector of unknowns.} \end{array}$$

By introducing in addition the definitions

$$\begin{array}{ll} \sigma_0^2 & \dots \text{ a priori variance} \\ \underline{\Sigma} & \dots \text{ covariance matrix,} \end{array}$$

the cofactor matrix of observations is

$$\underline{Q}_\ell = \frac{1}{\sigma_0^2} \underline{\Sigma}, \quad (9.93)$$

and

$$\underline{P} = \underline{Q}_\ell^{-1} \quad (9.94)$$

is the weight matrix. Assuming  $n$  observations and  $u$  unknown parameters leads to a design matrix  $\underline{A}$  comprising  $n$  rows and  $u$  columns. For  $n > u$ , the system (9.92) is redundant (overdetermined) and, in general, nonconsistent because of observational errors or noise. To assure consistency, the noise vector  $\underline{n}$  is added to the vector of observations and Eq. (9.92) converts to

$$\underline{\ell} + \underline{n} = \underline{A} \underline{x}. \quad (9.95)$$

The solution of this system becomes unique by the least squares principle  $\underline{n}^T \underline{P} \underline{n} = \text{minimum}$ . The application of this minimum principle on the observation equations (9.95) leads to the normal equations

$$\underline{A}^T \underline{P} \underline{A} \underline{x} = \underline{A}^T \underline{P} \underline{\ell} \quad (9.96)$$

with the solution

$$\underline{x} = (\underline{A}^T \underline{P} \underline{A})^{-1} \underline{A}^T \underline{P} \underline{\ell} \quad (9.97)$$

which can be simplified to

$$\underline{x} = \underline{G}^{-1} \underline{g} \quad (9.98)$$

where  $\underline{G} = \underline{A}^T \underline{P} \underline{A}$  and  $\underline{g} = \underline{A}^T \underline{P} \underline{\ell}$ .

The cofactor matrix  $\underline{Q}_x$  follows from  $\underline{x} = \underline{G}^{-1} \underline{A}^T \underline{P} \underline{\ell}$  by the covariance propagation law as

$$\underline{Q}_x = (\underline{G}^{-1} \underline{A}^T \underline{P}) \underline{Q}_\ell (\underline{G}^{-1} \underline{A}^T \underline{P})^T \quad (9.99)$$

and reduces to

$$\underline{Q}_x = \underline{G}^{-1} = (\underline{A}^T \underline{P} \underline{A})^{-1} \quad (9.100)$$

by substituting  $\underline{Q}_\ell = \underline{P}^{-1}$ .

#### *Sequential adjustment*

Assume a partitioning of the observation model (9.95) into two subsets:

$$\underline{\ell} = \begin{bmatrix} \underline{\ell}_1 \\ \underline{\ell}_2 \end{bmatrix} \quad \underline{n} = \begin{bmatrix} \underline{n}_1 \\ \underline{n}_2 \end{bmatrix} \quad \underline{A} = \begin{bmatrix} \underline{A}_1 \\ \underline{A}_2 \end{bmatrix} \quad (9.101)$$

Using the first set only, a preliminary solution  $\underline{x}_{(0)}$  can be calculated according to (9.97) and (9.100) by

$$\begin{aligned} \underline{x}_{(0)} &= (\underline{A}_1^T \underline{P}_1 \underline{A}_1)^{-1} \underline{A}_1^T \underline{P}_1 \underline{\ell}_1 = \underline{G}_1^{-1} \underline{g}_1 \\ \underline{Q}_{x_{(0)}} &= (\underline{A}_1^T \underline{P}_1 \underline{A}_1)^{-1} = \underline{G}_1^{-1} \end{aligned} \quad (9.102)$$

Provided that there is no correlation between the two subsets of observations, the weight matrix

$$\underline{P} = \begin{bmatrix} \underline{P}_1 & \underline{0} \\ \underline{0} & \underline{P}_2 \end{bmatrix} \quad (9.103)$$

is a block-diagonal matrix. The matrix  $\underline{G}$  and the vector  $\underline{g}$  for the adjustment of the full set of observations result from adding the corresponding matrices and vectors for the two subsets:

$$\begin{aligned} \underline{G} &= \underline{A}^T \underline{P} \underline{A} = (\underline{A}_1^T \underline{P}_1 \underline{A}_1 + \underline{A}_2^T \underline{P}_2 \underline{A}_2) = \underline{G}_1 + \underline{G}_2 \\ \underline{g} &= \underline{A}^T \underline{P} \underline{\ell} = (\underline{A}_1^T \underline{P}_1 \underline{\ell}_1 + \underline{A}_2^T \underline{P}_2 \underline{\ell}_2) = \underline{g}_1 + \underline{g}_2 \end{aligned} \quad (9.104)$$

If the change of the preliminary solution  $\underline{x}_{(0)}$  due to the additional observation set  $\underline{\ell}_2$  is denoted as  $\Delta \underline{x}$ , then

$$(\underline{G}_1 + \underline{G}_2) (\underline{x}_{(0)} + \Delta \underline{x}) = \underline{g}_1 + \underline{g}_2 \quad (9.105)$$

is the appropriate formulation of the adjustment. This equation can be slightly rearranged to

$$(\underline{G}_1 + \underline{G}_2) \Delta \underline{x} = \underline{g}_1 + \underline{g}_2 - (\underline{G}_1 + \underline{G}_2) \underline{x}_{(0)} \quad (9.106)$$

where the right side, cf. Eq. (9.102), can be simplified because of the relation  $\underline{g}_1 - \underline{G}_1 \underline{x}_{(0)} = \underline{0}$  so that

$$(\underline{G}_1 + \underline{G}_2) \Delta \underline{x} = \underline{g}_2 - \underline{G}_2 \underline{x}_{(0)} \quad (9.107)$$

results. Resubstituting  $\underline{g}_2$  and  $\underline{G}_2$  from (9.104) yields

$$(\underline{G}_1 + \underline{G}_2) \Delta \underline{x} = \underline{A}_2^T \underline{P}_2 \underline{\ell}_2 - \underline{A}_2^T \underline{P}_2 \underline{A}_2 \underline{x}_{(0)} \quad (9.108)$$

or

$$(\underline{G}_1 + \underline{G}_2) \Delta \underline{x} = \underline{A}_2^T \underline{P}_2 (\underline{\ell}_2 - \underline{A}_2 \underline{x}_{(0)}) \quad (9.109)$$

and

$$\Delta \underline{x} = (\underline{G}_1 + \underline{G}_2)^{-1} \underline{A}_2^T \underline{P}_2 (\underline{\ell}_2 - \underline{A}_2 \underline{x}_{(0)}) \quad (9.110)$$

or, finally,

$$\Delta \underline{x} = \underline{K} (\underline{\ell}_2 - \underline{A}_2 \underline{x}_{(0)}) \quad (9.111)$$

where

$$\underline{K} = (\underline{G}_1 + \underline{G}_2)^{-1} \underline{A}_2^T \underline{P}_2 \quad (9.112)$$

Note that the term  $\underline{A}_2 \underline{x}_{(0)}$  in (9.111) can formally be considered as prediction for the observations  $\underline{\ell}_2$ .

The change  $\Delta \underline{Q}$  with respect to the preliminary cofactor matrix  $\underline{Q}_{x_{(0)}}$  is obtained from the relation

$$\underline{G} \underline{Q}_x = (\underline{G}_1 + \underline{G}_2) (\underline{Q}_{x_{(0)}} + \Delta \underline{Q}) = \underline{I} \quad (9.113)$$

where  $\underline{I}$  denotes the unit matrix. This equation is reformulated as

$$(\underline{G}_1 + \underline{G}_2) \Delta \underline{Q} = \underline{I} - (\underline{G}_1 + \underline{G}_2) \underline{Q}_{x_{(0)}} \quad (9.114)$$

and, since  $\underline{G}_1 \underline{Q}_{x_{(0)}} = \underline{I}$ , this reduces to

$$(\underline{G}_1 + \underline{G}_2) \Delta \underline{Q} = -\underline{G}_2 \underline{Q}_{x_{(0)}} \quad (9.115)$$

or

$$\Delta \underline{Q} = -(\underline{G}_1 + \underline{G}_2)^{-1} \underline{G}_2 \underline{Q}_{x_{(0)}} \quad (9.116)$$