PPE 1 A bifurcation diagram.[1]

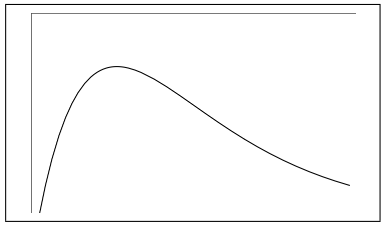

Figure 1: A humped graph.

Not all plants show the monotone seeding-yield relation depicted in Figure 1.1.1 of the course notes. In fact it may happen that at higher densities a dramatic collaps of the yield occurs. And a similar phenomenon has been observed for the so-called stock-recruitment relation for fisheries, where it is usually attributed to cannibalism. To investigate what kind of dynamics may result from such a humped graph, we shall study

$$x(n+1) = ax(n)e^{-x(n)} \tag{1}$$

Exercise 1.
Determine the steady states and their stability character, in dependence on the parameter a.

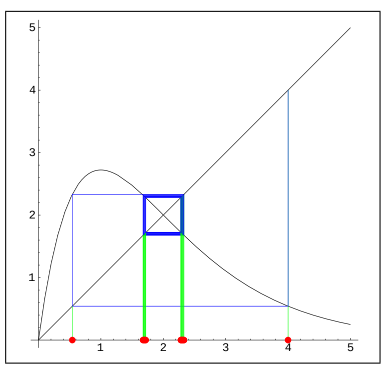

Figure 2: A 2-cycle.

For a moment we fix $a = 7.4$. We choose some initial value x_0 and start cobwebbing. The first 15 steps are shown in Figure 2.

Exercise 2.
Determine our choice for x_0 from the figure.

From the picture we conclude that we have found a *2-cycle*, which is an attractor.

[1]PPE stands for *Pen and Paper Exercises*.

Exercise 3.
Provide a precise definition for yourself of *2-cycle* and then generalise the definition to a n-cycle.

Exercise 4.
(i). Derive a recurrence relation for looking two years ahead.
Hint: use function composition.

(ii). Verify that the period two cycle shows up as two fixed points for this relation.

We are now going to apply the results of Exercise 1. and Figure 2. in order to sketch a first version of the *bifurcation diagram* that corresponds to Equation 1. The bifurcation diagram visualises the long time behaviour of $x(n)$, depending on the value of a. That is, on the *horizontal*-axis we let a or better, $ln(a)$ vary. For a fixed value of $ln(a)$ we put a dot for every value of $x(n)$ that appears for, say, $50 < n < 100$. Thus, the *vertical*-axis displays the long time values for $x(n)$.

Exercise 5.
(i). How many different values do you expect in the long run for $x(n)$, when $ln(a) < 2$? And how many when $ln(a)$ is slightly bigger than 2?

(ii). Sketch a bifurcation diagram that reflects your answers in part i).

For larger values of $ln(a)$, the recurrence relation will display chaotic behaviour. We will investigate this, using the computer, in the Mathematica session. This will enable us to extend the bifurcation diagram to larger values of $ln(a)$.

***Exercise 6.**
Suppose we had started with the following equation.

$$x(n+1) = ax(n)e^{-bx(n)} \tag{2}$$

Verify that b can be eliminated by using bx as a new variable.

Scaling x is, since x is number/area, like chosing a different unit to measure area and clearly the choice of unit can influence magnitude but not dynamics. Conclude that therefore our analysis also holds for the more general situation as modelled in Equation 2.