

PPE 10 A gambling game.

Two players A and B start out with a certain amount of money, say, respectively, s_A and s_B dollars, where s_A and s_B are integers. We define $s = s_A + s_B$. The play consists of flipping a fair coin (here "fair" means that both heads and tails have probability $\frac{1}{2}$ of coming up). When heads comes up, A pays one dollar to B, while when tails comes up B pays one dollar to A. The play continues till one of the two is cleaned out (and then we call the other one the winner). Natural questions are: with what probability will A win? How long (in the sense of "how many throws") is the game expected to last?

We use a Markov chain set-up to find answers. In particular we want to investigate how the answers depend on the two parameters s_A and s .

In the following we consider $s = s_A + s_B$ as a given quantity and we use the asset of A as the state variable (which consequently can take the values $0, 1, 2, \dots, s$). We adopt the thought convention that the coin is being flipped even after the game has ended, but then with nothing happening whatever the outcome. This redundancy allows us to use standard terminology, like absorbing state.

Let $p_i(n)$ with $i = 0, 1, \dots, s$ be the probability that A owns $\$i$ after n coin flippings.

Exercise 1.

(i). $p_i(n+1) = Mp_i(n)$ for some transition matrix M . Write down M for the case $s = 3$.

Also in the following questions ii)-vi) you may formulate your answers taking $s = 3$. There are two absorbing states.

(ii). Which are these?

(iii). What are the corresponding eigenvectors of M ?

(iv). Assume A starts out with $\$2$. What is the corresponding starting vector $p_i(0)$?

(v). This is a fair game, so the *expected* asset of each player should remain unchanged. Pinpoint the mathematical reflection of this idea. (See Conclusion 10.2.12 of the syllabus).

(vi). Compute the probability that A wins in terms of its starting capital s_A (so for $s_A = 1$ and for $s_A = 2$). Hint: study (the answer to) Exercise 10.2.5.vi.