PPE 2 Taylor series.

Exercise 1.
Verify two of the following Taylor approximations in $x = 0$.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + O(x)^{10}$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + O(x)^{10}$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + O(x)^{10}$$
$$\tan x = x + \frac{x^3}{3} + \frac{2\,x^5}{15} + \frac{17\,x^7}{315} + \frac{62\,x^9}{2835} + O(x)^{10}$$
$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + O(x)^{10}$$
$$\arcsin x = x + \frac{x^3}{6} + \frac{3\,x^5}{40} + \frac{5\,x^7}{112} + \frac{35\,x^9}{1152} + O(x)^{10}$$
$$\arccos x = \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3\,x^5}{40} - \frac{5\,x^7}{112} - \frac{35\,x^9}{1152} + O(x)^{10}$$
$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6} - \frac{x^7}{7} - \frac{x^8}{8} - \frac{x^9}{9} + O(x)^{10}$$

Exercise 2.
The exponential function can be defined by

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{1}$$

Verify, using this definition, that the first of the following properties holds.

1. $(\exp(x))' = \exp(x)$ and $\exp(0) = 1$
2. $\exp(x + y) = \exp(x)\exp(y)$ and $\exp(1) = e$
3. $\exp(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$

Hint: by convention $a^0 = 1$, for all a, even when $a = 0$.
Bonus: you can also try to prove the second and third property. You should not try to do this rigorousive, just indicate why they are true. Also each of these "properties" gives an alternate, but equivalent, definition of the exponential function. Again, indicate why this is true.

Exercise 3.
Prove that the following function is $\mathcal{C}^1$ but not $\mathcal{C}^2$.

$$f(x) = \int_0^x |\, t \,|\, dt \tag{2}$$

Is this f $O(h^2)$ near 0?