PPE 3 Maxima and minima.

Exercise 1.
Suppose that the number of bacteria in a culture at time t is given by

$$N = 5000(25 + te^{-t/20})$$

(i). Find the largest and smallest number of bacteria in the culture during the time interval $0 \leq t \leq 100$.

(ii). At what time during the time interval in part (i) is the number of bacteria decreasing most rapidly?

Exercise 2.
Define

$$f(x) = x^3 + ax$$

Determine the extrema of f on the interval [-1,1] for all values of a.

A *polynomial* in one variable is an expression of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

n is called the *degree* of the polynomial. The $a_0, \ldots, a_n$ are called the *coefficients*. So $x^3 + ax$ is a polynomial of degree three, but $x^3 + \sin(x)$ is not a polynomial.

Exercise 3.
(i). Show that the values of $p(-\infty_)$ and $p(\infty)$ are fully determined by the degree of $p(x)$ together with the sign of a_n.

(ii). Determine by inspection whether $p(x) = 3x^4 + 4x^3$ has any extrema. If so, find them and state where they occur.

In the following exercise we analyse what we can say about a stationary point $\tilde{x}$ when $D^2 f(\tilde{x}) = 0$ and f is at least $m+1$ times differentiable.. For the moment assume that the stationary point is situated in the origin, that is $\tilde{x} = 0$. Now suppose we are in the following situation:

$$f(\tilde{x}) = Df(\tilde{x}) = D^2 f(\tilde{x}) = D^3 f(\tilde{x}) = \cdots = D^{m-1} f(\tilde{x}) = D^m f(\tilde{x}) = 0,$$

but $D^{m+1} f(\tilde{x}) \neq 0$.

Exercise 4.
(i). Show that locally near 0

$$f(x) = \frac{1}{(m+1)!} D^{m+1} f(\tilde{x}) x^{m+1} + O(x^{m+2})$$

(ii). Show that there is a saddle point at $\tilde{x}$ if m is even.

(iii). Show that $f(x)$ has a local maximum at $\tilde{x}$ if m is odd and $f^{m+1}(\tilde{x}) < 0$ and a local minimum if m is odd and $f^{m+1}(\tilde{x}) > 0$.

(iv). Are these statements still true when $\tilde{x} \neq 0$?