PPE 4 Integration.

Exercise 1.
In this exercise we prove that the area of a circle C of radius r equals πr^2 by using integration.

(i). Give a geometrical reasoning that leads to the expression **??**.

$$\text{area}(C) \quad = \quad 4\int_0^r \int_0^{\sqrt{r^2-y^2}} 1 \; dxdy \qquad (1)$$

(ii). Show that

$$\int_0^r \int_0^{\sqrt{r^2-y^2}} 1 \; dxdy \quad = \quad \frac{\pi}{4} r^2$$

Hint: compare this exercise to the derivation of the volume of water in a trough in notebook 2.

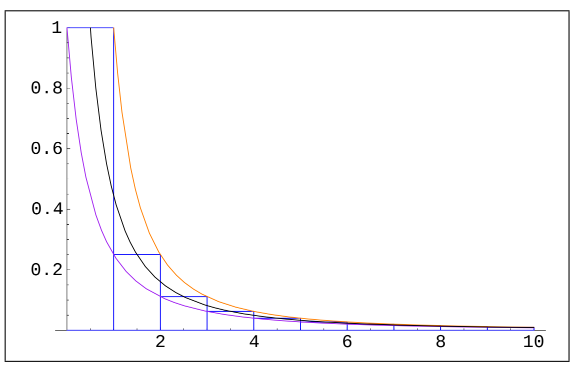

Figure 1: The first terms of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ together with a majorating, a minorating and a third function.

Exercise 2.
In this exercise we do it the other way around! The idea behind, for example, the midpoint rule is that we approximate the area of a complicated integral by something easy, in this case a number of rectangles. And this works good when we only have a finite number of rectangles. Now consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

by this we mean the quantity

$$s \quad := \quad 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots$$

It is easy to show that s is some real number, but it turns out quite hard to actually compute s. But we can find an upper and a lower bound for s as follows.

(i). Show that there exists a natural step-function $f = f(x)$ such that

$$s \quad = \quad \int_0^\infty f(x)dx \quad = \quad 1 + \int_1^\infty f(x)dx$$

Our strategy now will be to find easy to integrate positive functions $g(x)$ and $h(x)$ such that $g(x) \leq f(x) \leq h(x)$ for $x \in [1, \infty)$. Then we can conclude that

$$1 + \int_{x=1}^\infty g(x)dx \quad \leq \quad s \quad \leq \quad 1 + \int_{x=1}^\infty h(x)dx$$

(ii). Show that the choice $h(x) = \frac{1}{x^2}$ satisfies the requirement.

(iii). Determine an upper bound for s.

(iv). Find a suitable function $g(x)$ and use it for finding a lower bound.

This gives us a first idea on the value of s. But we can even find a really nice approximation.

(v). Find a function $r(x)$ by translating $g(x)$ such that s is the midpoint approximation of $r(x)$ on $[1, \infty]$ w.r.t. a suitable width h.

(vi). Show that

$$1 + \int_1^\infty r(x)dx \geq s$$

(vii). Compute $\int_1^\infty r(x)dx$.

By now it is time to give away that $s = \frac{\pi^2}{6}$. This means that your last approximation is a really good one!

Assume that you have a calculator that can compute square roots but has no key that produces an approximation of π.

(viii). Determine a method in order to compute digits of π.