PPE 8 A 2-dimensional recurrence relation.

Exercise 1.
Consider the matrix

$$M = \sigma \begin{pmatrix} 8 & 16 \\ 7 & 32 \end{pmatrix}$$

Determine the characteristic equation, and, by solving it, the eigenvalues of M as a function of the parameter σ.

We define a linear recurrence in terms of M.

$$x(n) = Mx(n-1) \tag{1}$$

Exercise 2.
Determine the behaviour of the solutions of Equation 1 in terms of σ. More precisely, for which values of σ do all solutions decay to zero and for which values of σ do some (or all?) solutions grow beyond bound?

Now we rescale $x(n)$ at every iteration. That is, we change the recurrence relation into

$$x(n) := \frac{Mx(n-1)}{\|Mx(n-1)\|}$$

Here $\|.\|$ denotes the L_1 norm. The L_1 norm of a vector (a, b) is defined as

$$\|(a, b)\| := |a| + |b|$$

The point is that now $\|x(n)\| = 1$ for all n, so $x(n)$ can neither decay to zero nor grow to infinity.

Exercise 3.
(i). Prove that $\|k \cdot x\| = |k|\|x\|$, for every scalar k and every vector x.

(ii). Prove that eigenvectors corresponding to a positive eigenvalue are fixed points of this new recurrence relation. What happens if we start with an eigenvector corresponding to a negative eigenvalue?

(iii). Prove that

$$x(n) := \frac{M^n x(0)}{\|M^n x(0)\|}$$

(iv). Prove that $x(n)$ is independent of σ, when $\sigma > 0$.